MATHEMATICS

B.Sc Part II (Paper III)

Topic - Sequences and their convergence

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Definition:

(i) <u>Sequence</u>: If, to each positive integer 1, 2, 3, ---- there corresponds a definite real number u_n, then the numbers $u_1, u_2, u_3, \ldots, u_n \ldots$ are said to form a sequence.

A sequence of elements $u_1, u_2, \ldots u_n \ldots$ is denoted by (u_n) or by $\{u_n\}$.

A sequence is said to be finite or infinite according as the number of elements is finite or infinite.

We shall consider only the infinite sequences.

<u>EXAMPLES</u>, (i) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$

(ii) $2, 4, 6, \ldots, 2n, \ldots$

(ii) <u>Monotonic Sequences</u>: If $u_{n+1} \geq u_n$ for every value of n, then the sequence $\{u_n\}$ is said to be monotone increasing.

If $u_{n+1} > u_n$ for every value of n, then $\{u_n\}$ is said to be monotone increasing in the stricter sense.

<u>EXAMPLE</u> (i) The sequence $1, 1, 3, 3, 5, 5, \ldots$ is monotone increasing

(ii) The sequence $1, 3, 5, 7, \ldots (2n-1), \ldots$ is monotone increasing sequence in the stricter sense.

If $u_{n+1} \leq u_n$ for every value of n, then the sequence $\{u_n\}$ is said to be monotone decreasing.

If $u_{n+1} \leq u_n$ for every value of n, then the sequence $\{u_n\}$ is said to be monotone decreasing in the stricter sense.

<u>EXAMPLE</u> (i) The sequence $-1, -1, -3, -3, -5, -5, \ldots$ is monotone decreasing.

(ii) The sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{n}, \ldots$ is monotone decreasing sequence in the stricter sense.

(iii) Limits of a sequence.

If corresponding to any positive number ϵ that we may choose, no matter how small, there is a positive integer m such that, for every integer n greater than or equal to m.

$$|u_n - l| < \epsilon, \text{ i.e., } l - \epsilon < u_n < l + \epsilon,$$

where l is a fixed number, then l is called the limit of the sequence $\{u_n\}$.

This can also be expressed by writing

$$\lim_{n\to\infty} u_n = l, \text{ or simply } \lim u_n = l.$$

If corresponding to any positive number N that we may choose no matter how great, there is a positive integer m, such that for every integer n greater than or equal to m, $u_n > N$, then the sequence $\{u_n\}$ is said to tend to infinity.

(iv) convergent sequence: The sequence $\{u_n\}$ is said to be convergent if the limit l of the sequence is a finite and definite number,

i.e., have been given a positive number ϵ, however small, there exists a positive integer m depending on ϵ such that $|u_n - l| < \epsilon$ for $n \geq m$, then $\{u_n\}$ is said to be convergent to the limit l, a definite number.

If $l = 0$, i.e., $u_n \to 0$ as $n \to \infty$, then $\{u_n\}$ is called a null sequence.

(V) Divergent sequence. The sequence $\{u_n\}$ is divergent if there exists a positive number N, however large, corresponding to which there is a positive integer m such that $u_n > N$ for all values of $n \geq m$.

Theorem 1. Prove that a convergent sequence determines its limit uniquely.

Proof: Let $\{u_n\}$ be a convergent sequence.

If possible, let l and l' be its two limits.

Since $\{u_n\}$ is convergent, we have a positive integer m depending on ϵ, an arbitrary positive number however small, such that

$$|u_n - l| < \frac{\epsilon}{2}, \text{ for all } n \geq m$$

and

$$|u_n - l'| < \frac{\epsilon}{2}, \text{ for all } n \geq m.$$

$$\text{Now } |l - l'| = |(u_n - l') - (u_n - l)|$$
$$\leq |u_n - l'| + |u_n - l|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \text{ for all } n \geq m.$$

But ϵ is arbitrary positive number however small.

Hence ultimately $l = l'$. Thus the limit is unique.

Theorem 2. Prove that every convergent sequence is bounded.

Proof: Let the sequence $\{a_n\}$ be convergent. So it must tend to a limit, say l.

Then, by the definition of limit of a sequence, given $\epsilon > 0$, however small, there exists a positive integer m, such that

$$|a_n - l| < \epsilon, \text{ for all } n \geq m$$

or $\quad l - \epsilon < a_n < l + \epsilon$, for all $n \geq m$.

Let k and K be the least and the greatest of

$$a_1, a_2, a_3, \cdots a_{m-1}, l - \epsilon, l + \epsilon.$$

Then $k \leq a_n \leq K$, for all $n \in N$.

Hence the sequence $\{a_n\}$ is bounded.

Theorem 3. Prove that a monotone increasing sequence tends to its upper bound.

Proof: Let $\{a_n\}$ be a monotone increasing sequence whose (least) upper bound is M. Two cases arise:

(i) M is finite; (ii) M is infinite

When M is finite: By definition of least upper bound

(i) $a_n \leq M$ for all n. ——— (1)

(ii) $a_n + \epsilon > M$ for at least one n. Let this be true for $n = m$.

Then $a_m + \epsilon > M$; i.e., $a_m > M - \epsilon$, ---- (2)
But the sequence is monotone increasing. So
$a_n \geq a_m$ if $n \geq m$. -- (3)
$\therefore$ From (2) and (3), $a_n > M - \epsilon$ for all $n \geq m$.
Again from (1), $a_n \leq M$ for all n. So $a_n < M + \epsilon$ for all n and hence also for $n \geq m$.
$\therefore$ Combining these, $M - \epsilon < a_n < M + \epsilon$ for $n \geq m$.
$\therefore$ $|a_n - M| < \epsilon$ for $n \geq m$.
$\therefore$ $\{a_n\}$ tends to M.
When M is infinite: Then from (2), $a_m > M - \epsilon = N$ (where N is arbitrarily large). But $a_n \geq a_m > N$ for $n \geq m$.
$\therefore$ $\{a_n\}$ tends to ∞ and so to the upper bound ∞. Hence the theorem.

Theorem 4 : Prove that a monotone decreasing sequence tends to its lower bound.

Proof. Let $\{a_n\}$ be a monotone decreasing sequence whose greatest lower bound is K. Two cases arise:
(i) K is finite; (ii) K is infinite.
When K is finite: By definition of greatest lower bound we have (i) $a_n \geq K$ for all n, ——— (1)
(ii) $a_n < K + \epsilon$ for at least one n. Let it be true $n = m$.
Then $a_m < K + \epsilon$. ——— (2)
But $\{a_n\}$ is a monotone decreasing sequence. So
$a_n \leq a_m$ for $n \geq m$ ——— (3)
From (2) and (3), $a_n < K + \epsilon$ for $n \geq m$.
and from (1), $a_n > K - \epsilon$ for all n and so for $n \geq m$.
Combining these, $K - \epsilon < a_n < K + \epsilon$ for $n \geq m$
$\therefore |a_n - K| < \epsilon$, for $n \geq m$.
Hence $\{a_n\}$ tends to the greatest lower bound K.
When K is infinite: If the lower bound is infinite. It must be negative and given any positive number G we have $a_n < -G$, for at least one value of n, let it be m.
So $a_m < -G$. But $a_n \leq a_m$, for $n \geq m$.
$\therefore a_n < -G$, for $n \geq m$. Hence $a_n \to \infty$ as $n \to \infty$.